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LETTER TO THE EDITOR

Upper critical dimension of the Kardar–Parisi–Zhang equation

J K Bhattacharjee

Department of Theoretical Physics, Indian Association for Cultivation of Science, Jadavpur, Calcutta 700 032, India

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Abstract. A recent mapping of the interface roughening problem to directed polymers by Lässig and Kinzelbech has shown that the upper critical dimension of the Kardar–Parisi–Zhang equation is less than or equal to four. By combining the mode-coupling technique with a small-α (roughening exponent) expansion, we show that the upper critical dimension is four. The validity of this conclusion is obviously limited by the applicability of the mode-coupling technique to the strong-coupling regime.

A much studied model of interface dynamics is the Kardar–Parisi–Zhang (KPZ) model [1–3] where the height $h(\boldsymbol{r}, t)$ of the interface above a D-dimensional substrate satisfies the equation of motion,

$$\frac{\partial h}{\partial t} = \nabla^2 h + \lambda(\boldsymbol{\nabla} h)^2 + \eta$$
$$\langle \eta(\boldsymbol{r}, t)\eta(\boldsymbol{r}', t')\rangle = 2D_0\delta^{(D)}(\boldsymbol{r} - \boldsymbol{r}')\delta(t - t'). \tag{1}$$

For $D < 2$, the model gives a rough interface for all values of λ. For $D > 2$, the consequences are less well understood. A rough phase exists for $\lambda > \lambda_c$ (a critical value) for $2 < D < D_c$, but cannot be obtained from perturbation theory.

This has led to the development of new techniques to deal with the strong-coupling problem. The task of a theory is to calculate the roughness exponent α. It is defined via the response function $G(k, \omega)$ and the correlation function $C(k, \omega)$

$$\delta^{(D)}(\boldsymbol{k} + \boldsymbol{k}')\delta(\omega + \omega')G(k, \omega) = \left\langle \frac{\delta h(\boldsymbol{k}, \omega)}{\delta\eta(\boldsymbol{k}', \omega')} \right\rangle$$
$$\delta^{(D)}(\boldsymbol{k} + \boldsymbol{k}')\delta(\omega + \omega')C(k, \omega) = \langle h(\boldsymbol{k}, \omega)h(\boldsymbol{k}', \omega')\rangle. \tag{2}$$

The response and correlation functions have the scaling form

$$G(k, \omega) = k^{-z} g\left(\frac{\omega}{k^z}\right) \tag{3a}$$

$$C(k, \omega) = k^{-D-2\alpha-z} f\left(\frac{\omega}{k^z}\right). \tag{3b}$$

The dynamic exponent z sets the true scale. A rough surface corresponds to $\alpha > 0$.

One expects $\alpha = 0$ at $D = D_c$. The task of the theory is to predict α as a function of D as well as the critical value D_c. The value of D_c has been controversial.

The mode-coupling theories [4–6], which are those that are capable of producing a value of D_c, have yielded different results under different handlings, including one [7] with $D_c = \infty$. Recently, Lässig and Kinzelbach [8] have adopted a completely different point of view. They map the KPZ problem onto directed polymers with quenched disorder and find $D_c \leqslant 4$. Our aim is to show that in fact the mode-coupling theories are in agreement with this result. In this letter, based on the work of Bouchaud and Cates [5] and the mode-coupling–perturbative-renormalization-group correspondence in dynamic critical phenomenon, we propose a new technique for obtaining α and D_c. Our idea is to perform a perturbation theory around $\alpha = 0$. We will show that a loop expansion can be converted into an expansion in powers of α. The response function in the scaling limit is

$$G^{-1}(k,\omega) = -\mathrm{i}\omega + \Sigma(k,\omega) \tag{3}$$

with $\Sigma(k,\omega) = k^z\sigma(\omega/k^z)$. The zero-frequency self-energy (in other words, the relaxation rate) is

$$\Sigma(k,0) = \Gamma k^z \tag{4}$$

and the zero-frequency correlation function can be written as

$$C(k,0) = Dk^{-2\alpha-D-z}. \tag{5}$$

Our contention is that the universal amplitude ratio $\Gamma^2/D\lambda^2$ can be written from the diagrammatics of $G^{-1}(k,\omega)$ as

$$\Gamma^2/D\lambda^2 = \sum_{n=1}^{\infty} I_n(D,\alpha)\alpha^{n-1} \tag{6}$$

and from the diagrammatics of $C(k,\omega)$ as

$$\Gamma^2/D\lambda^2 = \sum_{n=1}^{\infty} J_n(D,\alpha)\alpha^{n-2}. \tag{7}$$

The integer n labels the loops and thus equations (6) and (7) stand for a loop-wise expansion for the universal number. We have used $\alpha + z = 2$ which follows from Galilean invariance and hence must always be respected. Equating $\Gamma^2/D\lambda^2$ from equations (6) and (7) yields α as a function of D.

We explain the technique by working with $\Sigma(k,\omega)$. The single-loop expression is

$$\Sigma(k,\omega) = \lambda^2 \int \frac{\mathrm{d}\omega'}{2\pi}\frac{\mathrm{d}^D p}{(2\pi)^D}(\boldsymbol{p}\cdot\boldsymbol{q})(\boldsymbol{k}\cdot\boldsymbol{p})C(p,\omega')G(q,\omega-\omega') \qquad \boldsymbol{p}+\boldsymbol{q}=\boldsymbol{k}. \tag{8}$$

The frequency convolution is the tricky affair and gives rise to our first key observation. If one is at $D = D_c$, where $\alpha = 0$, the line shape is Lorentzian and an expansion about $\alpha = 0$ implies an expansion of the line shape about a Lorentzian form. With

$$C(k,\omega) = Dk^{-D-2\alpha}[G(k,\omega) + G^*(k,\omega)]$$

we can, correct to first-order deviations from the Lorentzian shape, write equation (8) after some algebra as

$$\Sigma(k,\omega) = \lambda^2 \int \frac{\mathrm{d}^D p}{(2\pi)^D}\frac{(\boldsymbol{p}\cdot\boldsymbol{q})(\boldsymbol{k}\cdot\boldsymbol{p})}{p^{2\alpha+D}}\frac{D}{-\mathrm{i}\omega + \Sigma(q,\mathrm{i}\Sigma(q)) + \Sigma(p,\mathrm{i}\Sigma(p))}. \tag{9}$$

Writing $\Sigma(k,\mathrm{i}\Sigma(k)) = \Gamma k^z$ and $\Sigma(k,0) = \Gamma_0 k^z$,

$$\frac{\Gamma^2}{D\lambda^2} = \int \frac{\mathrm{d}^D p}{(2\pi)^D}\frac{(\boldsymbol{1}\cdot\boldsymbol{p})[\boldsymbol{p}\cdot(\boldsymbol{1}-\boldsymbol{p})]}{p^{2\alpha+D}[(\Gamma_0/\Gamma) + p^z + |\boldsymbol{1}-\boldsymbol{p}|^z]}. \tag{10}$$

The distinction between Γ and Γ_0 is $\mathrm{O}(\alpha)$.

At this point we make the second key observation. The region of momentum space which dominates the integral in equation (10) for $\alpha \to 0$ is the region $p \gg 1$.

Making the appropriate approximates in the integrand, the right-hand side of equation (10) is

$$\frac{\alpha}{2D}\int_{p>1}\frac{\mathrm{d}^D p}{p^{D+\alpha}}$$

and thus we can write

$$\frac{\Gamma^2}{D\lambda^2} = \frac{\alpha}{2D}\int_{p>1}\frac{\mathrm{d}^D p}{(2\pi)^D}\frac{1}{p^{D+\alpha}} + \left[\int\frac{\mathrm{d}^D p}{(2\pi)^D}\frac{(\mathbf{1}\cdot\boldsymbol{p})[\boldsymbol{p}\cdot(\mathbf{1}-\boldsymbol{p})]}{p^{2\alpha+D}((\Gamma_0/\Gamma)+p^z+|\mathbf{1}-\boldsymbol{p}|^z)}\right.$$
$$\left. -\frac{\alpha}{2D}\int\frac{\mathrm{d}^D p}{(2\pi)^D}\frac{1}{p^{D+2\alpha}}\right]. \tag{11}$$

The first term on the right-hand side of equation (11) yields D^{-1} and the terms in the square brackets have to be evaluated as $\alpha \to 0$. This yields the correction to the first term.

The integrand in the square brackets being required for $\alpha \to 0$, it is permissible to set $\Gamma_0 = \Gamma$ in evaluating the integral to this order of accuracy. The $\mathrm{O}(\alpha)$ term so obtained takes care of the self-energy, insertion-type, two-loop graphs. To complete the $\mathrm{O}(\alpha)$ term, one needs the leading-order contribution from the vertex-correction-type, two-loop graphs. For leading order these are not necessary. To leading order, equation (11) yields

$$\frac{\Gamma^2}{D\lambda^2} = \frac{S_D}{(2\pi)^D}\frac{1}{2D} \tag{12}$$

where S_D is the surface area of a D-dimensional sphere.

Turning now to the correlation function, the single-loop, self-consistent answer is

$$C(k,\omega) = \frac{\lambda^2}{2}|G(k,\omega)|^2\int\frac{\mathrm{d}^D p}{(2\pi)^D}\frac{\mathrm{d}\omega'}{2\pi}C(p,\omega')C(k-p,\omega-\omega'). \tag{13}$$

Carrying out manipulations identical to those for $\Sigma(k,\omega)$ and equating the zero-frequency parts, we obtain

$$\frac{\Gamma^2}{D\lambda^2} = \frac{1}{2}\int\frac{\mathrm{d}^D p}{(2\pi)^D}\frac{[(\mathbf{1}-\boldsymbol{p})\cdot\boldsymbol{p}]^2}{p^{D+2\alpha}|\mathbf{1}-\boldsymbol{p}|^{D+2\alpha}}\frac{1}{[p^{2-\alpha}+|\mathbf{1}-\boldsymbol{p}|^{2-\alpha}]}. \tag{14}$$

Once again, we extract the high momentum ($p \gg 1$) part, and

$$\frac{\Gamma^2}{D\lambda^2} = \frac{1}{4}\int_{p>1}\frac{\mathrm{d}^D p}{(2\pi)^p}\frac{1}{p^{2D+3\alpha-2}} + \mathrm{O}(\alpha, D-2)$$
$$= \frac{1}{4}\frac{S_D}{(2\pi)^D}\frac{1}{D-2+3\alpha} + \text{higher order terms}. \tag{15}$$

Equating $\Gamma^2/D\lambda^2$ from equations (12) and (15),

$$D = 2D - 4 + 6\alpha \qquad \text{or} \qquad \alpha = \frac{4-D}{6}. \tag{16}$$

It is clear from the above result that α vanishes at $D = 4$, which is consequently the upper critical dimension for the problem. It is interesting to note that at $D = 1$ our formula gives the known exact result. This could, however, be accidental.

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